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Seminormality and Root Closure in Polynomial Rings and Algebraic Curves

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Let D be an integral domain with identity and let K be the quotient field of D. Then D is said to be *root closed* if whenever $\alpha \in K$ with $\alpha^n \in D$ for some positive integer n, then $\alpha \in D$. The domain D is called (2, 3)-*closed* if whenever $\alpha \in K$ with $\alpha^2, \alpha^3 \in D$, then $\alpha \in D$ and D is called F-*closed* if whenever $\alpha \in K$ with $n\alpha \in D$ for some positive integer n and $\alpha^2, \alpha^3 \in D$, then $\alpha \in D$. Clearly, if D is root closed, then D is (2, 3)-closed and if D is (2, 3)-closed, then D is F-closed.

The property of being root closed arose in Sheldon's work [7] on how changing D changes the quotient field of $D[[X]]$. As for (2, 3)-closure, its significance is due to the fact that an integral domain D is (2, 3)-closed if and only if D is seminormal if and only if $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1, \ldots, X_n])$, where Pic denotes the Picard group [4, Theorem 1]. Concerning F-closure, it is shown in [1] that the domain D is F-closed if and only if $D[X]$ is D-invariant—i.e., if and only if whenever $(D[X])\,[X_1, \ldots, X_n] \simeq_D S[Y_1, \ldots, Y_n]$, then S is D-isomorphic to $D[X]$. As noted in the preceding paragraph, these properties are related, although in their original non-arithmetic forms, they did not appear to be.

It is a consequence, albeit a deeply hidden one, of the proof of Theorem 1 of [4] that if D is (2, 3)-closed, so is $D[X]$. In this paper, we give arithmetic proofs that each of these three properties respects polynomial extension. In fact, we do this in greater generality and our approach is a unifying one in that we show that if an integral domain D is "n-root closed", then so is $D[X]$. The notion of "n-root closed" is then utilized to delineate the arithmetic distinction between normality and seminormality for algebraic curves. Recall that Bombieri [2] has shown that the geometric distinction between a normal curve and a seminormal curve is that the seminormal curve may have "ordinary" singular points. We prove that the coordinate ring of an irreducible algebraic curve over an algebraically closed field K is integrally closed if and only if it is (2, 3)-closed plus n-root closed for some n prime to the characteristic of K.

In the process of showing that our results cannot be extended to arbitrary reduced rings, we give a negative answer to a question implicit in [5, Ex.11,

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p. 100]. Namely, we exhibit a reduced ring T equal to its own total quotient ring, but having the property that $T[X]$ is not integrally closed.

All our rings are commutative and contain an identity element. The symbol "X" will always denote an indeterminate. If the nilradical of a ring R is (0), then we shall say that R is ***reduced***. A reduced ring of Krull dimension zero will be called ***absolutely flat***. Another name for such rings is "von Neumann regular ring."

Results

Let R be a ring with total quotient ring T. If $n > 1$ is a positive integer, then we shall say that R is ***n-root closed*** if whenever $\alpha \in T$ with $\alpha^n \in R$, then $\alpha \in R$. Note that R is root closed if and only R is n-root closed for each positive integer $n > 1$. Although our main interest lies in the case when R is an integral domain and T is its field of quotients, our arguments can be given quite naturally in a more general setting. Thus, let R be a subring of the ring S. For a positive integer $n > 1$, we shall say that R is ***n-root closed*** in S if whenever $\alpha \in S$ with $\alpha^n \in R$, then $\alpha \in R$. The statements "R is root closed in S", "R is (2, 3)-closed in S", and "R is F-closed in S" are similarly defined. Our first task is to prove stability of n-root closure under passage to polynomial extension.

THEOREM 1. *Let R be a subring of the ring S. If R is n-root closed in S, then $R[X]$ is n-root closed in $S[X]$.*

Proof. Suppose that R is a subring of S and that n is an integer which factors as $n = lm$. It is easy to check that R is n-root closed in S if and only if R is both l-root closed and m-root closed in S. From this it follows that R is n-root closed in S if and only if R is p-root closed in S for each prime divisor p of n. Consequently, it is sufficient to prove Theorem 1 in the case where n is prime.

The key to the proof is the concept of a minimal counterexample. For a fixed integer n and a fixed pair of rings $R \subseteq S$ such that R is n-root closed in S, we mean by a ***counterexample*** a polynomial $f \in S[X]$ with $f^n \in R[X]$, but $f \notin R[X]$. (Our goal is to show counterexamples don't exist.) A ***minimal counterexample*** is a counterexample $f = \sum_{j=0}^{m} a_j X^j$ having

(i) smallest degree m among all counterexamples, and

(ii) longest initial string $a_0, \ldots, a_{i-1}$ of coefficients in R among all counterexamples of degree m.

Note that if $f = \sum_{j=0}^{d} a_j X^j$ is a counterexample, then since $f^n \in R[X]$, $a_0{}^n \in R$ and hence $a_0 \in R$. Thus non-empty initial strings as in (ii) exist and have length $i \geqslant 1$. The first coefficient of f which is not in R is a_i, which we call the *critical coefficient* of f.

If counterexamples exist, then minimal counterexamples also exist. Moreover, if f is a minimal counterexample, $r \in R$, and $rf \notin R[X]$, then rf is also a minimal counterexample. For we certainly have $(rf)^n \in R[X]$, so that rf is a counterexample, and since $\deg(rf) \leqslant \deg(f)$, we must have $\deg(rf) = m$. And if $a_0, \ldots, a_{i-1}$ is the initial string of coefficients of f which are in R, with $a_i \notin R$, then $ra_0, \ldots, ra_{i-1} \in R$ so that rf has initial string of length at least i. Since f is a minimal counterexample, we see that rf must have initial string of length exactly i with critical coefficient ra_i. We shall refer to the minimal counterexample rf obtained from f in this fashion as a *normalization* of f. This discussion establishes the following key fact:

PRINCIPLE. If f is a minimal counterexample with critical coefficient a_i, then $ra_i \in R$ for $r \in R$ implies $rf \in R[X]$.

An element $b \in R$ such that $bf \in R[X]$, where $f \in S[X]$, we call a *booster* for f.

CLAIM 1. *If f is a minimal counterexample with $r^k f \in R[X]$ for some $r \in R$ and $k \geqslant 1$, then there is a normalization $f' = r^e f$, $e \geqslant 0$, such that $rf' \in R[X]$.*

To see this, observe that since $f \notin R[X]$ and $r^k f \in R[X]$, there is an integer e', $1 \leqslant e' \leqslant k$, such that $r^{e'} f \in R[X]$, but $r^{e'-1} f \notin R[X]$. Then let $e = e' - 1$.

Using Claim 1 we can now show that any minimal counterexample $f = \sum_{j=0}^{m} a_j X^j$ can be normalized to a minimal counterexample $f' = \sum_{j=0}^{m} a_j' X^j$ such that either n or a_0' is a booster for f'. For suppose a_i is the critical coefficient of f. The coefficient of X^i in $f^n \in R[X]$ has the form $na_0^{n-1} a_i$ plus a sum of terms $a_{j_1} \cdots a_{j_n}$ with $j_k < i$ for $k = 1, \ldots, n$. Since $a_0, \ldots, a_{i-1} \in R$, each of these products $a_{j_1} \cdots a_{j_n}$ is in R. Thus $na_0^{n-1} a_i \in R$. We have already observed that $a_0 \in R$, so $na_0^{n-1} \in R$. Since $na_0^{n-1} a_i \in R$, the Principle tells us that in fact $na_0^{n-1} f \in R[X]$. If $a_0^{n-1} f \notin R[X]$, then $f' = a_0^{n-1} f$ is a normalization of f such that $nf' \in R[X]$ and so n is a booster for f'. If $a_0^{n-1} f \in R[X]$, then by Claim 1, there is a normalization $f' = a_0{}^e f$ such that $a_0 f' \in R[X]$. In this case the constant term for f' is $a_0' = a_0^{e+1}$, so $a_0' f' = a_0{}^e (a_0 f') \in R[X]$, and the constant term a_0' is a booster for f' as desired.

We say that a minimal counterexample f is of *type I* if n is a booster for f, while f is of *type II* if its constant term is a booster for f. What we have shown to this point, then, is that if counterexamples exist, then there are minimal counterexamples of type I or there are minimal counterexamples of type II.

We come now to the crux of the argument. We claim that if f is a minimal counterexample having $b \in R$ as a booster, then f can be normalized to f' so that $bf', \ldots, b(f')^{n-1} \in R[X]$. In fact, we will show that we can normalize f to an f' with the property that $b\pi \in R$ for any product π of $n - 1$ or fewer coefficients of f'. Observe that since $bf \in R[X]$ by the definition of booster, we have $ba_j \in R$ for each coefficient a_j of f. Now assume that we have normalized to obtain a minimal counterexample $f = \sum_{j=0}^{m} a_j X^j$ such that

$$ba_{j_1} \cdots a_{j_M} \in R \text{ for all choices of coefficients } a_{j_1}, \ldots, a_{j_M} \text{ and all } M \leqslant N, \text{ where } N < n - 1. \tag{1}$$

We proceed by induction on N.

Examine the coefficient of X^{ni} in $f^n \in R[X]$; it has the form

$$a_i^n + \sum a_{l_1} \cdots a_{l_n} \in R \qquad (l_1 + \cdots + l_n = ni), \tag{2}$$

where each term $a_{l_1} \cdots a_{l_n}$ has at least one index $l_t < i$. Then $a_{l_t} \in R$ and $a_{l_1} \cdots \hat{a}_{l_t} \cdots a_{l_n}$ is a product of $n - 1$ coefficients of f. Write $n - 1 = Nq + s$, where $0 \leqslant s < N$. Since $a_{l_1} \cdots \hat{a}_{l_t} \cdots a_{l_n}$ may be grouped into q products of N coefficients and one product of s coefficients, it follows from (1) that $b^{q+1}a_{l_1} \cdots \hat{a}_{l_t} \cdots a_{l_n} a_i^{N-s} \in R$; and since $a_{l_t} \in R$ we get

$$b^{q+1}a_{l_1} \cdots a_{l_n}a_i^{N-s} \in R. \tag{3}$$

Thus from (2) we deduce that $b^{q+1}a_i^{n+N-s} \in R$. Now $n + N - s = (qN + s + 1) + N - s = (q + 1) N + 1$. Hence $(ba_i^N)^{q+1}a_i \in R$. From (1) we have $r = ba_i^N \in R$. Then by the Principle $r^{q+1}f \in R[X]$; and by Claim 1 there is a normalization $f' = r^e f$ with booster r. Since f' is a multiple of f by an element in R, b is still a booster for f' and in fact (1) holds for the coefficients of f'.

Now if as usual we denote by a_j' the coefficients of f', $b(a_i')^{N+1} = ba_i'(a_i')^N = ba_i'(r^e a_i)^N = (ba_i^N)a_i' r^{eN} = (ra_i')r^{eN} \in R$, since r is a booster for f'. Thus $(ba_i'^N)a_i \in R$ and from (1) $ba_i'^N \in R$, so by the Principle $ba_i'^N f' \in R[X]$. This shows that $(ba_i'^N)a_{j_1}' \in R$ for any coefficient a_{j_1}' of f'. Now by (1) again, $ba_i'^{N-1}a_{j_1}' \in R$ and since $(ba_i'^{N-1}a_{j_1}')\, a_i \in R$, the Principle tells us that $ba_i'^{N-1}a_{j_1}'f' \in R[X]$. Hence $ba_i'^{N-1}a_{j_1}'a_{j_2}' \in R$ for any coefficients a_{j_1}', a_{j_2}' of f'. Continuing in this manner we eventually find that $ba_i'a_{j_1}' \cdots a_{j_N}' \in R$ for any coefficients $a_{j_1}', \ldots, a_{j_N}'$ of f'. By (1), $ba_{j_1}' \cdots a_{j_N}' \in R$ and using the Principle again we get $ba_{j_1}' \cdots a_{j_N}'f \in R[X]$, whence $ba_{j_1}' \cdots a_{j_N}'a_{j_{N+1}}' \in R$ for any coefficients $a_{j_1}', \ldots, a_{j_{N+1}}'$ of f'. This completes the inductive step in our proof that if f is a minimal counterexample with booster b, then f has a normalization f' with the property that $b\pi \in R$ for any product π of $n - 1$ or fewer coefficients of f'.

Now suppose f is a minimal counterexample of type I. Then n is a booster for f and we have just seen that f has a normalization f' such that $nf', \ldots, n(f')^{n-1} \in R[X]$. On the other hand, if f is of type II with constant term a_0, f has a normalization $f' = rf$ for some $r \in R$, such that $a_0 f', \ldots, a_0(f')^{n-1} \in R[X]$. Since $a_0' = ra_0$, we have as well $a_0'f', \ldots, a_0'(f')^{n-1} \in R[X]$. Thus we have shown that if counterexamples exist, then there is either a minimal counterexample $f \in S[X]$ such that $nf, \ldots, nf^{n-1} \in R[X]$, or there is a minimal counterexample f with constant term a_0 such that $a_0 f, \ldots, a_0 f^{n-1} \in R[X]$.

Let f be a counterexample of either of the two types just mentioned. Write $f(X) = a_0 + Xg(X)$. Then $X^n g(X)^n = (f - a_0)^n = f^n - \binom{n}{1}a_0 f^{n-1} + \cdots +$

$(-1)^{n-1}\binom{n}{n-1}a_0^{n-1}f + (-1)^n a_0^n$. Recalling the fact that $a_0 \in R$, that $f^n \in R[X]$, and that we can assume n is prime so that n divides $\binom{n}{i}$ for $i \neq 0, n$, we see that if f is either of the two types of counterexample, the right hand side of the equation is in $R[X]$, whence $X^n g(X)^n \in R[X]$. It follows that $g(X)^n \in R[X]$, and since $f \notin R[X]$ and $a_0 \in R$, we must have $g \notin R[X]$. But then g is a counterexample of degree less than m, a contradiction. Therefore counterexamples do not exist, and the proof is complete.

In a similar vein we have the following result whose proof is much easier.

PROPOSITION 1. *Let R be a subring of the ring S. Then*

(a) *If R is* (2, 3)*-closed in S, then $R[X]$ is* (2, 3)*-closed in $S[X]$, and*

(b) *If R is F-closed in S, then $R[X]$ is F-closed in $S[X]$.*

Proof. We again use the notion of a counterexample. In case (a) a counterexample is a polynomial $f \in S[X]$ such that $f^2, f^3 \in R[X]$, but $f \notin R[X]$. In case (b) a counterexample is a polynomial $f \in S[X]$ such that $f^2, f^3 \in R[X]$ and for some positive integer n, $nf \in R[X]$, but $f \notin R[X]$. Note that in either case if f is a counterexample with constant term a_0, then $a_0 \in R$. Define minimal counterexamples in either case exactly as in the proof of Theorem 1. Observe that in either case the Principle still holds.

Now in case (a) or case (b) let f be any minimal counterexample with critical coefficient a_i. The coefficient of X^i in f^2 is in the form $2a_0a_i +$ (terms in R), whence $2a_0a_i \in R$. The coefficient of X^i in f^3 is in the form $3a_0^2a_i +$ (terms in R), whence $3a_0^2a_i \in R$. Using the Principle we find that $2a_0f \in R[X]$ and $3a_0^2f \in R[X]$.

Let $f(X) = a_0 + Xg(X)$. Then $X^2g(X)^2 = (f - a_0)^2 = f^2 - 2a_0f + a_0^2 \in R[X]$, and $x^3g(X)^3 = (f - a_0)^3 = f^3 - 3a_0f^2 + 3a_0^2f - a_0^3 \in R[X]$ since f^2, $f^3 \in R[X]$. It follows that $g^2, g^3 \in R[X]$; and in case (b) $ng \in R[X]$ follows from $nf \in R[X]$. Since $a_0 \in R$ and $f \notin R[X]$, we have $g \notin R[X]$. But then g is a counterexample of impossibly smaller degree in either case (a) or case (b). This shows that no counterexamples exist, so the proof is complete.

We shall summarize our results in Theorem 2, but as we now show, it is necessary first to confront the following problem. If S is a reduced ring equal to its own total quotient ring, is $S[X]$ integrally closed? For let R be a ring having total quotient ring T. By Theorem 1, if R is n-root closed (in T), then $R[X]$ is n-root closed in $T[X]$. Since

$$R[X] \subseteq T[X] \subseteq \text{total quotient ring of } R[X],$$

we see that $R[X]$ is n-root closed if and only if $T[X]$ is n-root closed. Consequently, for $R[X]$ to be n-root closed it is sufficient that $T[X]$ be integrally closed. We shall make use of this observation in the proof of Theorem 2.

It is known that if $T[X]$ is integrally closed (or even n-root closed for some n), then T must be reduced and consequently that R must be reduced. To see this, if $r \neq 0$ is such that $r^2 = 0$, then $r^n = 0$ and so $(r/(1 + X))^n \in T[X]$, but

$r/(1+X) \notin T[X]$ for if $r/(1+X) \in T[X]$, then $r = (1+X)f(X)$ for some $f \in T[X]$. By specializing X to -1, we see that $r = 0$.

We now present an example to show that T can be reduced without $T[X]$ being n-root closed.

EXAMPLE 1. This is an example of a reduced ring T equal to its own total quotient ring such that $T[X]$ is not F-closed. In particular, $T[X]$ is not integrally closed.

We first claim that it is sufficient to construct reduced rings $R \subseteq S$ of positive characteristic having elements b, $c \in R$ and $e \in S$ which satisfy the following conditions:

(1) $be \in R$

(2) $ec = 0$

(3) $e^n \in R$, for $n \geqslant 2$

(4) $\mathrm{Ann}_s(b, c) = \{s \in S \mid sb = 0 = sc\} = 0$

(5) there do not exist elements r, $s \in R$ with s a non-zero-divisor such that $(se - r)b = 0$.

Indeed, suppose we have R, S, b, c, e as above. Let T be the total quotient ring of R. By condition (4) the polynomial $b + cX$ is a non-zero-divisor in $R[X]$ [5, p. 348], and therefore also in $T[X]$, so the rational function $f(X) = be/(b + cX)$ is in the total quotient ring of $T[X]$. By condition (2), $(be)^n = (b + cX)^n e^n$, so $f(X)^n = e^n \in R$ for $n \geqslant 2$, using (3). In particular $f(X)^2$, $f(X)^3 \in T[X]$. Note that also $qf(X) = 0 \in T[X]$, where $q = \mathrm{char}\, R > 0$. But we cannot have $f(X) \in T[X]$, for if so we would have $be = bt$ for some $t \in T$. Write $t = r/s$ with r, $s \in R$ and s a non-zero-divisor in R. Then $(se - r)b = 0$, contradicting (5). Thus T is the desired example.

Now to construct R and S. Let k be a field of positive characteristic and let X, Y be indeterminates over k. Let Π be the set of irreducible polynomials in $k[X, Y]$ which have zero constant term, and let $\mathbb{Z}^+$ be the set of positive integers. Consider a set $\{Z_{\pi,i} \mid (\pi, i) \in \Pi \times \mathbb{Z}^+\}$ of indeterminates over $k[X, Y]$, and let $A = k[X, Y, \{Z_{\pi,i}\}]$. Let I be the ideal of A generated by $\{Z_{\pi,i} \cdot Z_{\pi',i'} \mid (\pi, i) \neq (\pi', i')\} \cup \{\pi(X, Y) \cdot Z_{\pi,i} \mid (\pi, i) \in \Pi \times \mathbb{Z}^+\}$. Let $B = k[X, Y, XZ_{Y,1}, Z_{Y,1}^2, Z_{Y,1}^3, \{Z_{\pi,i} \mid (\pi, i) \neq (Y, 1)\}] \subseteq A$, and let $J = I \cap B$. Then we set $R = B/J$, $S = A/I$, $b = \bar{X}$, $c = \bar{Y}$, $e = \bar{Z}_{Y,1}$, where the bar denotes residue class modulo I.

By construction, $R \subseteq S$, $\mathrm{char}\, R > 0$, $be \in R$, $ec = 0$, and $e^n \in R$ for $n \geqslant 2$. Thus conditions (1)–(3) are satisfied, so it remains to show that S is reduced and that (4) and (5) hold. In order to do so, observe that any polynomial $G(X, Y, Z) \in A$ is congruent modulo I to a polynomial of the form

$$F(X, Y, Z) = \alpha + f(X, Y) + \sum_{\pi,i} h_{\pi,i}(X, Y, Z_{\pi,i})\, Z_{\pi,i}, \qquad (*)$$

where $\alpha \in k$, $f(X, Y)$ has zero constant term, and $h_{\pi,i} \in k[X, Y, Z_{\pi,i}]$. (This is true because any mixed Z terms, $Z_{\pi,i} \cdot Z_{\pi',i'}$, occurring in $G(X, Y, Z)$ are already in I.) Hence each element of S can be realized as the residue class of a polynomial of the form in (*).

To see that S is reduced, consider an element $s = \overline{F(X, Y, Z)}$ with $F(X, Y, Z)$ as in (*). If $s^m = 0$, $F(X, Y, Z)^m \in I$, and by specializing all the variables $Z_{\pi,i}$ to 0, we get $(\alpha + f(X, Y))^m = 0$ in $k[X, Y]$, since I specializes to the zero ideal. Thus $\alpha = 0$ and $f(X, Y) = 0$, so that $F = \sum h_{\pi,i}(X, Y, Z_{\pi,i})Z_{\pi,i}$. Specializing all the $Z_{\pi,i}$'s to 0 except for Z_{π_0,i_0} maps I to the ideal of $k[X, Y, Z_{\pi_0,i_0}]$ generated by $\pi_0(X, Y)Z_{\pi_0,i_0}$, and hence transforms the relation $F^m \in I$ to

$$h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})^m Z^m_{\pi_0,i_0} \in (\pi_0(X, Y)\, Z_{\pi_0,i_0}).$$

Since $\pi_0(X, Y)$ is irreducible, this implies that $\pi_0(X, Y)$ divides $h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})$. But then $F \in I$, whence $s = 0$, proving that S is reduced.

To verify (4) it will suffice to show that $\operatorname{Ann}_S(b) \cdot \operatorname{Ann}_S(c) = 0$. For then $(\operatorname{Ann}_S(b) \cap \operatorname{Ann}_S(c))^2 \subseteq \operatorname{Ann}_S(b) \cdot \operatorname{Ann}_S(c) = 0$, and since S is reduced $\operatorname{Ann}_S(b, c) = \operatorname{Ann}_S(b) \cap \operatorname{Ann}_S(c) = 0$. In fact we will show that $\operatorname{Ann}_S(b)$ is generated by $\{\bar{Z}_{X,i} \mid i \in \mathbb{Z}^+\}$, whence by symmetry $\operatorname{Ann}_S(c)$ is generated by $\{\bar{Z}_{Y,i} \mid i \in \mathbb{Z}^+\}$. Since $\bar{Z}_{X,i} \cdot \bar{Z}_{Y,j} = 0$ for all i and j, (4) will follow. Thus let $s = \overline{F(X, Y, Z)}$ as in (*) and suppose that $sb = 0$. Then $F(X, Y, Z)X \in I$, and specializing all $Z_{\pi,i}$ to zero yields $(\alpha + f(X, Y))X = 0$ in $k[X, Y]$. Thus $\alpha = f(X, Y) = 0$ and $F = \sum h_{\pi,i}(X, Y, Z_{\pi,i})Z_{\pi,i}$. Specializing all $Z_{\pi,i}$ to zero except for Z_{π_0,i_0} yields $h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})Z_{\pi_0,i_0}X \in (\pi_0(X, Y)Z_{\pi_0,i_0})$, so $\pi_0(X, Y)$ divides $h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})X$. If $\pi_0(X, Y) \neq X$, then $\pi_0(X, Y)$ divides $h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})$ and $h_{\pi_0,i_0}(X, Y, Z_{\pi_0,i_0})Z_{\pi_0,i_0} \in I$. If $\pi_0(X, Y) = X$, then $h_{\pi_0,i_0}Z_{\pi_0,i_0} = h_{X,i_0}Z_{X,i_0} \in AZ_{X,i_0}$. It follows that $s \in (\{\bar{Z}_{X,i} \mid i \geqslant 1\})$. Since it is clear by construction that $\{\bar{Z}_{X,i} \mid i \geqslant 1\} \subseteq \operatorname{Ann}_S(b)$, we have $\operatorname{Ann}_S(b) = (\{\bar{Z}_{X,i} \mid i \geqslant 1\})$, as desired.

Before establishing (5) we need to know something about elements of S which are regular on R (i.e. do not annihilate non-zero elements of R). Namely, an element $s = \overline{F(X, Y, Z)}$, with F as in (*), is regular on R if and only if the constant term α is non-zero in k. To see this suppose first that $\alpha = 0$. Since $f(X, Y)$ has constant term zero, it has an irreducible factor $\pi(X, Y)$ with constant term zero. In the expression $F = f(X, Y) + \sum h_{\pi,i}Z_{\pi,i}$ there are only a finite number of non-zero terms, and so we may choose an integer N larger than all indices i appearing in the sum. Then by our construction of S, $\bar{Z}_{\pi,N} \cdot s = 0$, while $\bar{Z}_{\pi,N} \neq 0$. Being careful to choose $N \geqslant 2$ guarantees that $\bar{Z}_{\pi,N} \in R$. We leave it to the interested reader to show that if $\alpha \neq 0$, then s is in fact regular on S. (We won't use this implication in what follows.)

To verify (5) we now show that there do not exist elements s, $a \in S$ with s regular on R and $a \in \operatorname{Ann}_S(b)$ such that $se - a \in R$. (a plays the role of $se - r$ in (5)). For suppose such elements exist, and write $s = \bar{F}$ as in (*). By the preceding paragraph we know that since s is regular on R, $\alpha \neq 0$. Multiplying by α^{-1} does no harm, so we may assume that $\alpha = 1$. Our verification of (4) showed $\operatorname{Ann}_S(b) = (\{\bar{Z}_{X,i} \mid i \geqslant 1\})$, so we may write $a = \sum_i \bar{g}_i \cdot \bar{Z}_{X,i}$, with $g_i \in A$ for $i \geqslant 1$. Then the relation $se - a \in R$ is equivalent to

$$\left(1 + f(X, Y) + \sum h_{\pi,i} Z_{\pi,i}\right) Z_{Y,1} - \sum g_i Z_{X,i} \in B + I.$$

Specializing X, Y, and all $Z_{\pi,i}$'s except $Z_{Y,1}$ to zero sends I to zero and B to $k[Z_{Y,1}^2, Z_{Y,1}^3]$. Thus the above relation specializes to $(1 + h_{Y,1} Z_{Y,1}) Z_{Y,1} \in k[Z_{Y,1}^3, Z_{Y,1}^2]$, which is not true. This completes the verification of (5) and so establishes the existence of the example.

From the example and the remarks preceding it we see that some hypotheses on R and the total quotient ring T of R are required to assure the stability of n-root closure under polynomial ring formation. In particular, R must be reduced. An oft applicable assumption on T is that it be absolutely flat. This leads us to the following form of our main stability theorem.

THEOREM 2. *Suppose that R is a reduced ring whose total quotient ring T is an absolutely flat ring. In particular, R could be any integral domain or reduced noetherian ring. Then*

(i) *Let $n > 1$ be a positive integer. If R is n-root closed, so is $R[X]$.*

(ii) *If R is root closed, so is $R[X]$.*

(iii) *If R is (2, 3)-closed, so is $R[X]$.*

(iv) *If R is F-closed, so is $R[X]$.*

Proof. By the remarks preceding example 1, the proof amounts to showing that if S is an absolutely flat ring, then $S[X]$ is integrally closed. But this follows from the fact that $S[X]$ is a Bézout ring—that is, each finitely generated ideal of $S[X]$ is principal [6, p. 224]. More specifically, suppose that A is a Bézout ring with total quotient ring B. If $a/s \in B$, then $(a, s) = (t)$ for some non-zero-divisor t of A. Writing $a = ta'$ and $s = ts'$, we see that $a/s = a'/s'$ where $(a', s') = A$. Consequently, we can assume that each element of B can be written in the form a/s where $(a, s) = A$. We now give the standard argument. If a/s is integral over A, then

$$\left(\frac{a}{s}\right)^n + r_{n-1}\left(\frac{a}{s}\right)^{n-1} + \cdots + r_1\left(\frac{a}{s}\right) + r_0 = 0$$

for some $r_0, \ldots, r_{n-1} \in A$. Hence, $a^n + s r_{n-1} a^{n-1} + \cdots + r_0 s^n = 0$ from which it follows that $a^n \in (s)$. If P is a prime ideal of A containing s, then $a^n \in P$ and so $P \supseteq (a, s) = A$. Therefore, s is a unit and $a/s \in A$.

We completely change direction now and apply the notion of n-root closure in a geometric situation. The geometric distinction between a non-singular curve and a seminormal curve is that the seminormal one is allowed to possess ordinary singular points [2]. The arithmetic distinction problem becomes a meaningful one when interpreted as a question about coordinate rings. This problem amounts to distinguishing between an integrally closed domain and a (2, 3)-closed domain. For algebraic curves we can simultaneously treat the case of a root closed domain. This we do in the following theorem.

THEOREM 3. *Let D be the coordinate ring of an irreducible algebraic curve $\mathscr{C}$ over an algebraically closed field K. The following conditions are equivalent.*

(1) *D is integrally closed.*

(2) *D is root closed.*

(3) *D is (2, 3)-closed and D is n-root closed for some positive integer n prime to the characteristic of K. (If the characteristic of K is 0, then we can omit the phrase "prime to the characteristic of K.")*

Proof. It is clear that we have only to prove that (3) implies (1). Thus, let D satisfy condition (3) but not condition (1) so that $\mathscr{C}$ is certainly seminormal [4, Theorem 1] but not normal. By [2], if p is a singular point of $\mathscr{C}$ then p is an ordinary n-fold point, where n is the dimension of the tangent space at p. Therefore, the integral closure $\bar{\mathcal{O}}_p$ of the local ring $\mathcal{O}_p$ of $\mathscr{C}$ at p has n distinct maximal ideals $\mathscr{M}_1, \ldots, \mathscr{M}_n$. Let L be the function field of $\mathscr{C}$. Then $\{V_i = (\bar{\mathcal{O}}_p)_{\mathscr{M}_i}\}_{i=1}^n$ is the set of valuation rings of L centered on $\mathscr{M}_p$, the maximal ideal of $\mathcal{O}_p$. Denote by v_i the valuation on L associated with V_i. Since K is algebraically closed, all places of L over K are rational and so $V_i = K + \mathscr{M}_i(\bar{\mathcal{O}}_p)_{\mathscr{M}_i}$. It follows that $\bar{\mathcal{O}}_p = K + \mathscr{M}_i$ for $i = 1, \ldots, n$. Because $\mathcal{O}_p$ is seminormal, $\mathscr{M}_p = \mathscr{M}_1 \cap \cdots \cap \mathscr{M}_n$ [4, Corollary 1]. Hence

$$\mathcal{O}_p = K + \mathscr{M}_p = K + (\mathscr{M}_1 \cap \cdots \cap \mathscr{M}_n).$$

By hypothesis, there is a positive integer t, prime to the characteristic of K such that D is t-root closed. It is easy to see that t-root closure is a local property and so $\mathcal{O}_p$ is t-root closed. Also, since t is prime to the characteristic of K, there is a non-trivial (i.e. $\neq 1$) t-th root of unity ξ in K. By the strong approximation theorem for independent valuations [3, p. 497], there exists an element $\alpha \in L$ such that

$$v_i(\alpha) \geqslant 0,\ i = 1, \ldots, n,\ v_i(\alpha - 1) \geqslant 1,\ i = 1, \ldots, n-1, \quad \text{and} \quad v_n(\alpha - \xi) \geqslant 1.$$

Thus, $\alpha \in \bar{\mathcal{O}}_p$ and moreover, since $x = \alpha - 1 \in \bar{\mathcal{O}}_p$, $x \in \mathscr{M}_1 \cap \cdots \cap \mathscr{M}_{n-1}$. Similarly, $y = \alpha - \xi \in \mathscr{M}_n$. Then

$$\begin{aligned} \alpha^t &= (1 + x)^t = 1 + (tx + \cdots + x^t) \\ &= (\xi + y)^t = 1 + (t\xi^{t-1}y + \cdots + y^t). \end{aligned}$$

Consequently, $tx + \cdots + x^t = t\xi^{t-1}y + \cdots + y^t \in \mathscr{M}_1 \cap \cdots \cap \mathscr{M}_n = \mathscr{M}_p$, whence $\alpha^t \in \mathcal{O}_p$ and so by the hypothesis of t-root closure $\alpha \in \mathcal{O}_p$.

Write $\alpha = a + z$, for some $a \in K$, $z \in \mathscr{M}_1 \cap \cdots \cap \mathscr{M}_n$. Because $\bar{\mathcal{O}}_p = K + \mathscr{M}_1$ is a direct sum, $a = 1$ and $z = x$. Likewise, $a = \xi$ and $z = y$. This yields the contradiction that $\xi = 1$ and the proof is complete.

We remark that in non-geometric situations it is not always possible to distinguish (2, 3)-closed domains from integrally closed domains merely by considering n-root closure. In fact, Exercise 6 of [5, p. 184] gives an example of a root closed domain which is not integrally closed.

A consequence of Theorem 3 is that we are able to geometrically separate the three conditions root closure, (2, 3)-closure, and F-closure. Thus let $\mathbb{C}$ be the field of complex numbers, let D_1 be the coordinate ring of the plane curve $\mathscr{C}_1$ over $\mathbb{C}$ determined by the equation $Y^2 = X^2 + X^3$, and let D_2 be the coordinate ring of the plane curve $\mathscr{C}_2$ determined over $\mathbb{C}$ by the equation $Y^2 = X^3$.

Since the only singularity of $\mathscr{C}_1$ is a node at the origin, $\mathscr{C}_1$ is seminormal [2] and so D_1 is (2, 3)-closed. However, D_1 is not root closed by Theorem 3.

Because $\mathscr{C}_2$ has a cusp at the origin, $\mathscr{C}_2$ is not seminormal [2] and so D_2 is not (2, 3)-closed [4, Theorem 1]. That D_2 *is* F-closed follows from the definition once we recall that $\mathbb{C} \subseteq D_2$.

We close by noting the curious fact that the domain D_1 is an example of an integral domain which is (2, 3)-closed, but which is neither 2-root closed nor 3-root closed.

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